Maps

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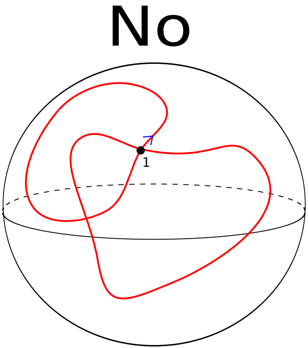
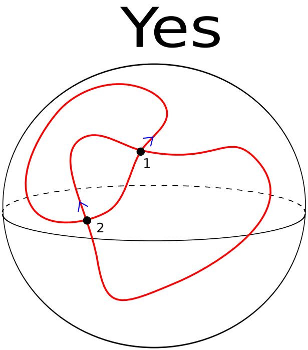
- Labels–
  - The vertices have distinct names
  - We choose a function assigning to each vertex one of its incident edges

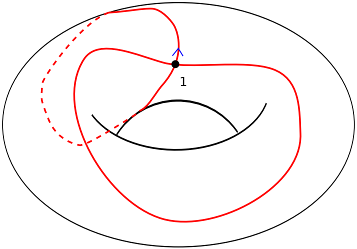
Edges cannot intersect





Faces must be discs

Yes



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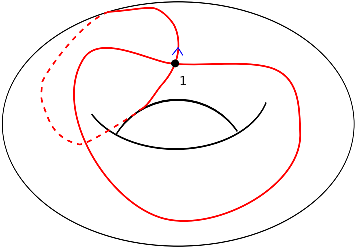
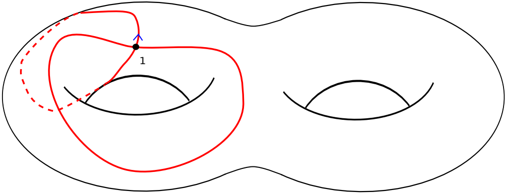
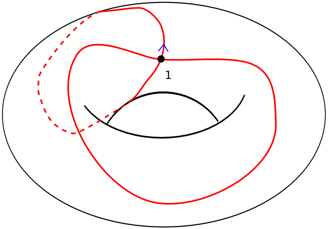


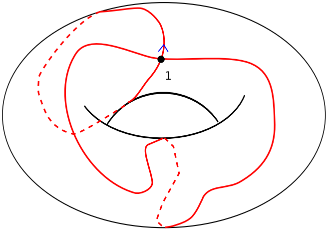
No



Dehn twist

These are the same map





How can $\log Z_n$ know about maps???

GUE covariances

Gaussian Unitary Ensemble:

$$dP_n(M) = \frac{1}{Z_n} e^{-\frac{1}{2}\operatorname{Tr} M^2}\, dM$$

$$\mathbb{E}[M_{ij} M_{kl}] = \delta_{i=l}\delta_{j=k}$$

Wick's lemma

$f_1, \ldots, f_{2m}$ are linear functionals on $\mathbb{R}^{n \times n}$.

$$\mathbb{E}[f_1 \ldots f_{2m}] = \sum_{\substack{w \in \text{Wick pairings} \\ \text{of } 1, \ldots, 2m}} \mathbb{E}[f_{w(1)} f_{w(2)}] \ldots \mathbb{E}[f_{w(2m-1)} f_{w(2m)}]$$

An example of a Wick pairing of $1, \ldots, 8$ is $\{\{1, 6\}, \{2, 5\}, \{3, 4\}, \{7, 8\}\}$.

Matrix Integrals and combinatorics

Here is a very brief hand wave at the connection between matrix integrals in map combinatorics.

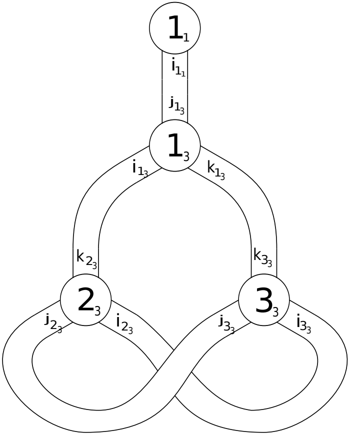
$$\mathbb{E}\left[\left(\mathrm{Tr}\, M^4\right)^p\right] = \mathbb{E}\left[\prod_{q=1}^{p} \sum_{i_q,j_q,k_q,l_q=1}^{n} M_{i_q,j_q} M_{j_q,k_q} M_{k_q,l_q} M_{l_q,i_q}\right]$$

$$= \sum_{\substack{i_1,j_1,k_1,\ldots \\ ,j_p,k_p,l_p=1}}^{n} \sum_{\substack{w\in\text{Wick pairings} \\ \text{of } 1,\ldots,4p}} \left(\begin{array}{c}\text{Product of quadratic} \\ \text{expectations given by} \\ w \text{ and index variables}\end{array}\right)$$

$$= \sum_{\substack{\text{4-valent fatgraphs} \\ \text{on } p \text{ vertices}}} n^{\text{Faces}}$$

A Wick pairing and the corresponding map

The Wick pairing $(i_{1_1}, j_{1_3}), (i_{1_3}, k_{2_3}), (k_{1_3}, k_{3_3}), (i_{2_3}, i_{3_3}), (j_{2_3}, j_{3_3})$ corresponds to a fatgraph and a map.



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